Construction of θ-Poincaré Algebras and their Invariants on $\mathcal{M}_\theta$

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In the present paper we construct deformations of the Poincaré algebra as representations on a noncommutative spacetime with canonical commutation relations. These deformations are obtained by solving a set of conditions by an appropriate ansatz for the deformed Lorentz generator. They turn out to be Hopf algebras of quantum universal enveloping algebra type with nontrivial antipodes. In order to present a notion of θ-deformed Minkowski space $\mathcal{M}_\theta$, we introduce Casimir operators and spacetime invariants for all deformations obtained.

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1 INTRODUCTION

An important issue of high energy physics is the unification of all physical interactions into a single renormalized quantum field theory. Most of the various approaches to this aim share the idea that both topics, unification and renormalization, should be addressed simultaneously by the introduction of a natural unit of length. The concept of a natural unit of length is almost as old as quantum field theory itself, since the cut-offs that ensure the finiteness of integrals over momentum space in ordinary perturbation theory actually correspond to nonlocal interactions that general relativity already suggests by the Planckian length

$$\lambda_p = (\frac{G\hbar}{c^3})^{\frac{1}{2}} \simeq 1.6 \times 10^{33} \text{cm}.$$

This fundamental unit of length marks the scale of energies and distances at which nonlocality of interactions has to appear and a notion of continuous spacetime becomes meaningless. Such nonlocal interactions have been introduced in various ways: the approaches to this matter range from rather mathematical concepts to fundamental physical motivations - depending on whether the aspect of renormalization or that of general relativity is emphasized. Apart from string theory the oldest and rather abstract path to the topic is that of noncommutative geometry. From the physical point of view this corresponds to a generalization of the quantization scheme by replacing the real valued coordinates of configuration space by a noncommutative algebra of hermitean operators. One of the earliest and most prominent examples is Snyder's work from the 1940s where he showed that, under the requirement of Lorentz covariant spacetime spectra, any introduction of a finite natural unit of length necessarily leads to a noncommutative algebra of coordinates [20]. Although Snyder's model exhibits intriguing properties that could lead to renormalization of self-energies and vacuum polarizations, it also features the main trouble noncommutative geometry has to deal with, namely the breaking of spacetime symmetries[2].

In the 1980s and early 1990s noncommutative spaces and their symmetries were investigated more systematically in the context of quantum groups which arose from the work of L.D. Faddeev on the inverse scattering method [8]. The first objects studied in quantum groups were deformed Lie algebras and groups such as $U_q(sl_2)$ of P.P. Kulish and N.Y. Reshetikhin [11] or compact quantum matrix groups such as $SU_q(2)$ of S.L. Woronowicz [22]. These quantum groups were identified to be Hopf algebras as E.K.Sklyanin showed for example for $U_q(sl_2)$ in [19]. Moreover V.G. Drinfeld and M. Jimbo found a whole class of one parameter deformations of semi-simple Lie algebras [7] being Hopf algebras of quantum universal enveloping algebra type. The study of representations always kept the contact to physical aspects. At the beginning of 1990s q-deformations of the Lorentz and Poincaré algebra on a q-Minkowski space [9] were obtained.

Despite their elegance and their mathematically rigorous construction these noncommutative spaces turn out to be far too complicated to construct field theories on them with a reasonable amount of effort. This is mainly due to the fact that the commutation relations of the corresponding quantum spaces are fully quadratic in the coordinates. This makes it impossible to define Moyal-Weyl star products in terms of exponential expressions. Prominent exception is the κ-Minkowski space [12] that allows a study of field theoretic aspects as for example in [15].

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[2]Snyder's construction leads to a de Sitter momentum space that brakes translational invariance. This problem was solved by Yang [23]. Further field theoretical aspects were studied by Gol'fand [10] later.

Parallel to the development of quantum groups, string theory blazed its trail to be the first serious attempt to unification and renormalization. In the last years open strings with homogenous magnetic background field [3] gave rise to so called brane world scenarios where the effective field theories live on noncommutative spaces with canonical commutation relations[3]. Seiberg-Witten map [18] and deformation quantization [14] opened the doorway to gauge theories on noncommutative spaces that even lead to noncommutative versions of the standard model of particles and the grand unified theory [16]. The main drawback in this latest approach is the absence of a scheme of quantization and of spacetime symmetries other than translation invariance. Note that noncommutative spaces with canonical commutation relations were also obtained by introducing the nonlocality by general relativistic arguments, where the involved constructions become covariant under Lorentz symmetries by imposing additional quantum conditions on the antisymmetric constant tensor [6]. However, since modern approaches to quantum gravity, as loop quantum gravity, suggest that not only the configuration space but also the symmetry algebra should be deformed [5], we consider an alternative way via quantum groups in the present paper.

We present here an attempt to join the area of quantum groups with that of field theories on θ-spacetime structures. We derive θ-Poincaré algebras as representations on the noncommutative spacetime algebra with canonical commutation relations. These deformations turn out to be Hopf algebras of quantum universal enveloping algebra type.

As a generalization of the present results we also sketch how one could possibly obtain deformed symmetry algebras to any *given* noncommutative space. In the traditional approach of quantum groups the algebraic properties of quantum spaces are determined by the deformation applied to the symmetry algebra. Here we fairly follow the opposite way - however this generalization is still work in progress.

In the first part of this paper we collect all requirements that any deformation of a symmetry algebra to any given noncommutative space has to obey. For θ - Poincaré algebras we restrict ourselves to the case of quantizations that are linear in the deformation parameters and show how deformations arise by the choice of an appropriate generating ansatz for the deformed Lorentz operator. We find continously many solutions that all turn out to be Hopf algebras. One of the solutions we present here was also found independently by alternative considerations by M. Dimitrijević et al. in the attempt to study concepts of derivatives on deformed coordinate spaces [21]. Furthermore the same solution was obtained by the authors of [4] using a suitable Drinfeld-twist.

Finally we work out explicit expressions for the Pauli-Lubanski vector and the configuration space invariant for all solutions. Thus we obtain a notion of θ-Minkowski space $\mathcal{M}_\theta$.

Our considerations incorporate the following notations and conventions. In general the field and quantum group theoretic aspects of our considerations orient themselves to the references [2] and [1]. We use latin and greek letters for indices of space and spacetime coordinates respectively

$$
\begin{aligned}
i, j, k, \ldots &\in \{1, 2, 3\} \\
\mu, \nu, \ldots &\in \{0, 1, 2, 3\}.
\end{aligned}
$$

Our presentation is independent of any specific choice of the signature for the metric tensor $\eta^{\mu\nu}$ in commutative Minkowski space $\mathcal{M}$.

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[3] A similar result was received at the beginning of the 1970s where the effective theory of charged particles in a homogenous electric field lead to the same noncommutative space [17].

2   THE POINCARÉ ALGEBRA AND ITS θ - DEFORMATIONS $U_\theta^\lambda(\mathfrak{p})$

In this section we derive θ-deformations of the Poincaré algebra $\mathfrak{p}$ as representations on noncommutative spacetime algebras $\mathfrak{X}_\theta$ with canonical commutation relations

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}.$$

We find continuously many solutions of quantum universal enveloping algebra type, $U_\theta^\lambda(\mathfrak{p})$, that are parametrized by real parameters $\lambda = (\lambda_1, \lambda_2)$.

The section contains three parts. In the first subsection we collect a set of three conditions that any deformation of the type $U_\theta(\mathfrak{p})$ has to satisfy. In the second part we present an ansatz for the operators $M^{\mu\nu}$ of the deformed Lorentz algebra that generates the desired solutions $U_\theta^\lambda(\mathfrak{p})$. Finally we conclude this section by the presentation of the Hopf structure of $U_\theta^\lambda(\mathfrak{p})$, i.e. we give explicit formulas for counits, coproducts and antipodes for all solutions that are considered here and give the proof of the Hopf algebra axioms.

In parallel, as mentioned in the introduction, we sketch a first scheme of a general method that shall provide the opportunity to derive deformations $U_h(\mathfrak{g})$ to any given noncommutative spacetime algebra $\mathfrak{X}_h$ with deformation parameter h. Hence the line of our arguments is drawn in terms of a general Lie algebra $\mathfrak{g}$ and we treat $U_\theta^\lambda(\mathfrak{p})$ on $\mathfrak{X}_\theta$ as an example.

We emphasize that the development of this method is still a work in progress. Here we merely want to draw the outline of our idea and show that already at this stage it can be applied successfully, as one of the solutions that we present here, $U_\theta^{(\frac{1}{2},0)}(\mathfrak{p})$, was also achieved recently by alternative approaches [21], [4]. Thus, many aspects that touch on to the presented scheme will be treated independently in our subsequent work.

2.1   CONDITIONS FOR DEFORMATIONS $U_\theta^\lambda(\mathfrak{p})$ AS REPRESENTATIONS OVER $\mathfrak{X}_\theta$

Since deformations $U_\theta^\lambda(\mathfrak{p})$ of the Poincaré algebra $\mathfrak{p}$ are of quantum universal enveloping algebra type we first clarify such notions as that of $U_h(\mathfrak{g})$ and that of representations on a given spacetime algebra $\mathfrak{X}_h$ in terms of a general Lie algebra $\mathfrak{g}$.

2.1 DEFINITION *A p-dimensional* Lie algebra *over the field* **K** *is a* **K**-*linear vector space endowed with a map*

$$[\,,\,]: \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

called bracket *with the following properties:*

$$\forall g, h, k \in \mathfrak{g}: \quad [g, h] = -[h, g] \;\; \textit{(Antisymmetry)}$$
$$[g + h, k] = [g, k] + [h, k] \;\; \textit{(Linearity)}$$
$$0 = [[g, h], k] + [[h, k], g] + [[k, g], h] \;\; \textit{(Jacobi-Identity)}$$

Linearity holds for both components.

Since it is a vector space, the Lie algebra $\mathfrak{g}$ has a p-dimensional basis $(g)_{i \in I}$ with $I = \{1 \ldots p\}$. Hence the bracket $[\,,\,]$ can be expressed as a linear combination of the basis elements in terms of the Lie algebra's structure constant $c_{ij}^k \in \mathbf{K}$. For all $i, j, k \in I$ we have then

$$[g_i, g_j] = ic_{ij}^k g_k.$$

4

Since direct sums and tensor products of vector spaces are vector spaces themselves, to any Lie algebra $\mathfrak{g}$ there exists the tensor or free algebra $\mathcal{T}(\mathfrak{g})$

$$\mathcal{T}(\mathfrak{g}) = \mathbf{K} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \ldots \oplus \mathfrak{g}^{j\otimes} \oplus \ldots.$$

that leads us directly to the constructive definition of a universal enveloping algebra.

2.2 Definition *If $\mathfrak{g}$ is a Lie algebra with bracket $[\,,\,]$ then the* universal enveloping algebra $U(\mathfrak{g})$ *of $\mathfrak{g}$ is the tensor algebra $\mathcal{T}(\mathfrak{g})$ devided by the two-sided ideal $\mathcal{I}$*

$$U(\mathfrak{g}) = \frac{\mathcal{T}(\mathfrak{g})}{\mathcal{I}}$$

that is generated by relations

$$g_i \otimes g_j - g_j \otimes g_i - i c_{ij}^k g_k = 0.$$

The deformation of a Lie algebra $\mathfrak{g}$ is performed by quantizing its universal enveloping algebra that we denote by $U_h(\mathfrak{g})$. This is because the commutator bracket $\left[G_i \overset{\otimes}{,} G_j \right] = G_i \otimes G_j - G_j \otimes G_i$ for generators $(G_i)_{i \in I}$ of $U_h(\mathfrak{g})$ maps within $U_h(\mathfrak{g})$, i.e. the commutator in general is a linear combination in terms of the infinit dimensional basis of $U_h(\mathfrak{g})$ generated by $(G_i)_{i \in I}$. Thus $U_h(\mathfrak{g})$ becomes a Lie algebra with

$$
\begin{aligned}
[\,,\,] : U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g}) &\longrightarrow U_h(\mathfrak{g}) \\
[G_i, G_j] &\mapsto i C_{ij}(G_k, h).
\end{aligned}
\tag{1}
$$

In the further consideration, we omit the symbol of tensor multiplication. The quantum universal enveloping algebra is thus defined to be the free associative algebra generated by $(G_i)_{i \in I}$ that is divided by the ideal $\mathcal{I}_h$ generated by (1) such that for $h \to 0 : \mathcal{I}_h \to \mathcal{I}$ and consequently

$$U_h(\mathfrak{g}) \to U(\mathfrak{g}).$$

We adjourn the discussion concerning the potential change of the number of generators $(G_i)_{i \in I}$ under deformations and exclude such solutions for $U_h(\mathfrak{g})$. To be well defined in this way the multiplication in $U_h(\mathfrak{g})$ has to satisfy closure and associativity. This is the first condition expressed by the Jacobi-Identity for the functions $C_{ij}(G_k, h)$

Condition 1

$$
\begin{aligned}
0 &= [[G_i, G_j], G_k] + [[G_j, G_k], G_i] + [[G_k, G_i], G_j] \\
&= i[C_{ij}(G_l, h), G_k] + i[C_{jk}(G_l, h), G_i] + i[C_{ki}(G_l, h), G_j].
\end{aligned}
$$

We now apply Condition 1 to the example of $U_\theta(\mathfrak{p})$. The commutation relations of the Poincaré algebra $\mathfrak{p}$ are given by

$$
\begin{aligned}
[p^\mu, p^\nu] &= 0 \\
[m^{\mu\nu}, p^\rho] &= i\eta^{\mu\rho} p^\nu - i\eta^{\nu\rho} p^\mu \\
[m^{\mu\nu}, m^{\rho\sigma}] &= i\eta^{\mu\rho} m^{\nu\sigma} - i\eta^{\nu\rho} m^{\mu\sigma} + i\eta^{\nu\sigma} m^{\mu\rho} - i\eta^{\mu\sigma} m^{\nu\rho}.
\end{aligned}
\tag{2}
$$

For the case of canonical commutation relations the deformation is actually limited to the Lorentz algebra, such that the first relation of (2) is preserved. As generators for $U_\theta(\mathfrak{p})$ we

use momentum operators P^μ and Lorentz operators $M^{\mu\nu}$. We make the following ansatz for the commutation relations of the deformed Poincaré algebra

$$
\begin{aligned}
[P^\mu, P^\nu] &= 0 \\
[M^{\mu\nu}, P^\rho] &= i(\eta^{\mu\rho}P^\nu - \eta^{\nu\rho}P^\mu) + i\chi_\theta^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda}) \\
[M^{\mu\nu}, M^{\rho\sigma}] &= +i\eta^{\mu\rho}M^{\nu\sigma} - i\eta^{\nu\rho}M^{\mu\sigma} + i\eta^{\nu\sigma}M^{\mu\rho} - i\eta^{\mu\sigma}M^{\nu\rho} \\
&\quad +i\phi_\theta^{\mu\nu\rho\sigma}(P^\gamma, M^{\kappa\lambda}).
\end{aligned}
\tag{3}
$$

The function $\phi_\theta^{\mu\nu\rho\sigma}(P^\gamma, M^{\kappa\lambda})$ is antisymmetric in the first and second pair of indices and has physical dimension 1. The function $\chi_\theta^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda})$ is antisymmmetric in the first pair of indices and has physical dimension length^{-1}. Inserting the commutation relations (3) into the Jacobi-Identities corresponding to Condition 1

$$
\begin{aligned}
0 &= [[P^\mu, P^\nu], M^{\rho\sigma}] + [[P^\nu, M^{\rho\sigma}], P^\mu] + [[M^{\rho\sigma}, P^\mu], P^\nu] \\
0 &= [[M^{\mu\nu}, M^{\rho\sigma}], P^\lambda] + [[M^{\rho\sigma}, P^\lambda], M^{\mu\nu}] + [[P^\lambda, M^{\mu\nu}], M^{\rho\sigma}] \\
0 &= [[M^{\mu\nu}, M^{\rho\sigma}], M^{\kappa\lambda}] + [[M^{\rho\sigma}, M^{\kappa\lambda}], M^{\mu\nu}] + [[M^{\kappa\lambda}, M^{\mu\nu}], M^{\rho\sigma}]
\end{aligned}
$$

gives the following relations for the functions $\phi_\theta^{\mu\nu\rho\sigma}(P^\gamma, M^{\kappa\lambda})$ and $\chi_\theta^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda})$

$$
\begin{aligned}
0 &= [P^\mu, \chi_\theta^{\rho\sigma\nu}] - [P^\nu, \chi_\theta^{\rho\sigma\mu}] \\
0 &= [P^\lambda, \phi_\theta^{\mu\nu\rho\sigma}] + [M^{\mu\nu}, \chi_\theta^{\rho\sigma\lambda}] - [M^{\rho\sigma}, \chi_\theta^{\mu\nu\lambda}] \\
0 &= i\left[\phi_\theta^{\mu\nu\rho\sigma}, M^{\kappa\lambda}\right] + i\left[\phi_\theta^{\rho\sigma\kappa\lambda}, M^{\mu\nu}\right] + i\left[\phi_\theta^{\kappa\lambda\mu\nu}, M^{\rho\sigma}\right] \\
&\quad +\eta^{\nu\rho}\phi_\theta^{\mu\sigma\kappa\lambda} - \eta^{\mu\rho}\phi_\theta^{\nu\sigma\kappa\lambda} + \eta^{\mu\sigma}\phi_\theta^{\nu\rho\kappa\lambda} - \eta^{\nu\sigma}\phi_\theta^{\mu\rho\kappa\lambda} \\
&\quad -\eta^{\sigma\kappa}\phi_\theta^{\mu\nu\rho\lambda} + \eta^{\rho\kappa}\phi_\theta^{\mu\nu\sigma\lambda} - \eta^{\rho\lambda}\phi_\theta^{\mu\nu\sigma\kappa} + \eta^{\sigma\lambda}\phi_\theta^{\mu\nu\rho\kappa} \\
&\quad +\eta^{\lambda\mu}\phi_\theta^{\kappa\nu\rho\sigma} - \eta^{\kappa\mu}\phi_\theta^{\lambda\nu\rho\sigma} + \eta^{\kappa\nu}\phi_\theta^{\lambda\mu\rho\sigma} - \eta^{\lambda\nu}\phi_\theta^{\kappa\mu\rho\sigma}.
\end{aligned}
\tag{4}
$$

Any pair of functions $\phi^{\mu\nu\rho\sigma}(P^\gamma, M^{\kappa\lambda})$ and $\chi_\theta^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda})$ solving these equations leads to well defined algebraic properties of $U_\theta(\mathfrak{p})$.

Since $U_\theta(\mathfrak{p})$ shall be a representation on $\mathfrak{X}_\theta$, we now consider the action of $G_i \in U_h(\mathfrak{g})$ on coordinates $x^\mu \in \mathfrak{X}_h$. To this purpose the symmetry algebra has to be enhanced by a coalgebra structure.

2.3 DEFINITION *The coalgebra structure on the **K**-vector space $U_h(\mathfrak{g})$ is given by the two linear operations counit ϵ and coproduct Δ. These are the maps*

$$
\begin{aligned}
\Delta : U_h(\mathfrak{g}) &\to U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g}) \\
G_i &\mapsto \Delta(G_i) = \sum G_{i(1)} \otimes G_{i(2)} \\
\epsilon : U_h(\mathfrak{g}) &\to K \\
G_i &\mapsto \epsilon(G_i)
\end{aligned}
$$

that obey the two coalgebra axioms of counit and coassociativity

$$
\begin{aligned}
(\epsilon \otimes id) \circ \Delta = \quad id \quad &= (id \otimes \epsilon) \circ \Delta \\
(\Delta \otimes id) \circ \Delta &= (id \otimes \Delta) \circ \Delta.
\end{aligned}
$$

2.4 DEFINITION *A bialgebra is **K**-vector space with algebra and coalgebra structure made compatible by demanding that the coproduct and counit are algebra homomorphisms*

$$\Delta(G_i G_j) = (\Delta G_i)(\Delta G_j), \qquad \Delta \mathbf{1} = \mathbf{1} \otimes \mathbf{1}$$
$$\epsilon(G_i G_j) = \epsilon(G_i)\epsilon(G_j), \qquad \epsilon(\mathbf{1}) = 1.$$

We now assume that $U_h(\mathfrak{g})$ is a bialgebra and consider its representations on $\mathfrak{X}_\mathfrak{h}$. Its associative multiplication for the coordinates $x^\mu, x^\nu \in \mathfrak{X}_h$ is given by the commutator

$$[x^\mu, x^\nu] = x^\mu x^\nu - x^\nu x^\mu = i\omega_h^{\mu\nu}(x^\rho).$$

Since the coordinates of $\mathfrak{X}_\mathfrak{h}$ are hermitean operators, the antisymmetric function $\omega_h^{\mu\nu}(x^\mu)$ is real valued.

2.5 DEFINITION *A representation is a pair $(\rho, \mathfrak{X}_h)$ containing the homomorphism*

$$\rho : U_h(\mathfrak{g}) \quad \rightarrow \quad gl(\mathfrak{X}_h)$$
$$G_i \quad \mapsto \quad \rho(G_i),$$

such that for $G_i, G_j \in U_h(\mathfrak{g})$ and $x^\mu, x^\nu \in \mathfrak{X}_h$

$$\rho(G_i G_j - G_j G_i - iC_{ij}(G_k, h))x^\mu = 0 \tag{5}$$

and

$$\rho(G_i)(x^\mu x^\nu - x^\nu x^\mu - i\omega_h^{\mu\nu}(x^\rho)) = 0 \tag{6}$$

are satisfied.

In other terms the algebraic structure of $U_h(\mathfrak{g})$ shall be represented in $gl(\mathfrak{X}_h)$ and these act as endomorphisms on $\mathfrak{X}_h$.
Introducing the left-action of $G_i \in U_h(\mathfrak{g})$ on coordinates $x^\mu \in \mathfrak{X}_h$ by

$$G_i \triangleright x^\mu = \rho(G_i)x^\mu,$$

products of coordinates are mapped according to

$$G_i \triangleright (x^\mu x^\nu) \quad = \quad \sum m\left(\left(G_{i(1)} \triangleright x^\mu\right) \otimes \left(G_{i(2)} \triangleright x^\nu\right)\right)$$
$$G_i \triangleright \mathbf{1} \quad = \quad \epsilon(G_i)\mathbf{1}.$$

The multiplication m is that of the coordinate algebra $\mathfrak{X}_h$. Before we continue to actually define the action, we precede with some important remarks. In contrary to the commutative case, the commutator $[G_i, x^\mu]$ is not an element within $\mathfrak{X}_h$. We rather find that

$$[G_i, x^\mu] \in \mathfrak{X}_h \otimes U_h(\mathfrak{g}).$$

Since $\rho(G_i) \in gl(\mathfrak{X}_h)$ the action cannot be defined by the commutator $[G_i, x^\mu]$. This is only possible in the limit of $h \rightarrow 0$.
Thus we define the action of $G_i \in U_h(\mathfrak{g})$ on the coordinates $x^\mu, x^\nu \in \mathfrak{X}_h$ with $\mathbf{1} \in \mathfrak{X}_h$ by

$$\rho(G_i)x^\mu = G_i \triangleright x^\mu := [G_i, x^\mu] \triangleright \mathbf{1},$$

where the action of coordinates $x^\mu \in \mathfrak{X}_h$ on the unit element is merely a multiplication: $x^\mu \triangleright \mathbf{1} = x^\mu \mathbf{1} = x^\mu$. This way relation (5) now reads

$$(G_i G_j - G_j G_i - iC_{ij}(G_k, h)) \triangleright x^\mu \quad = \quad 0$$
$$\Leftrightarrow ([G_i, [G_j, x^\mu]] - [G_j, [G_i, x^\mu]] - i[C_{ij}(G_k, h), x^\mu]) \triangleright \mathbf{1} \quad = \quad 0$$

and thus we obtain

CONDITION 2

$$[G_i, [G_j, x^\mu]] - [G_j, [G_i, x^\mu]] - i\, [C_{ij}(G_k, h), x^\mu] = 0.$$

Turning to relation (6) we compute

$$
\begin{aligned}
G_i \triangleright (x^\mu x^\nu - x^\nu x^\mu - i\omega_h^{\mu\nu}(x^\rho)) &= 0 \\
\Leftrightarrow ([[G_i, x^\mu], x^\nu] - [[G_i, x^\nu], x^\mu] - i\, [G_i, \omega_h^{\mu\nu}(x^\rho)]) \triangleright \mathbf{1} &= 0
\end{aligned}
\tag{7}
$$

and obtain

CONDITION 3

$$[[G_i, x^\mu], x^\nu] + [[x^\nu, G_i], x^\mu] + i\, [\omega_h^{\mu\nu}(x^\rho), G_i] = 0.$$

To read off the bialgebra structure of $U_h(\mathfrak{g})$ we assume that the coproduct is of the general form

$$\Delta(G_i) = G_i \otimes \mathbf{1} + \mathbf{1} \otimes G_i + \sum \xi_{i(1)} \otimes \xi_{i(2)}.$$

We apply this again to the relation (6)

$$
\begin{aligned}
0 =\ & (G_i \triangleright x^\mu) x^\nu + x^\mu (G_i \triangleright x^\nu) + \sum (\xi_{i(1)} \triangleright x^\mu)(\xi_{i(2)} \triangleright x^\nu) \\
& -(G_i \triangleright x^\nu) x^\mu - x^\nu (G_i \triangleright x^\mu) - \sum (\xi_{i(1)} \triangleright x^\nu)(\xi_{i(2)} \triangleright x^\mu) \\
& -iG_i \triangleright \omega_h^{\mu\nu}(x^\rho)
\end{aligned}
\tag{8}
$$

and compare with the computation (7) from above. We obtain for the coproduct

$$
\begin{aligned}
\sum (\xi_{i(1)} \triangleright x^\mu)(\xi_{i(2)} \triangleright x^\nu) &= ([G_i, x^\mu]\, x^\nu) \triangleright \mathbf{1} - ([G_i, x^\mu] \triangleright \mathbf{1})\, x^\nu \\
&= ([[G_i, x^\mu], x^\nu] + x^\nu\, [G_i, x^\mu]) \triangleright \mathbf{1} - ([G_i, x^\mu] \triangleright \mathbf{1})\, x^\nu.
\end{aligned}
\tag{9}
$$

This formula will be used in the next subsection to compute the coproduct for the deformed Lorentz generators $M^{\mu\nu}$. For instance we turn again to the example $U_\theta(\mathfrak{p})$. We make an ansatz for the commutator of $M^{\mu\nu}$ and coordinates $x^\rho \in \mathfrak{X}_\theta$. The corresponding relation for P^μ is of the classical form such that we have

$$
\begin{aligned}
[P^\mu, x^\rho] &= -i\eta^{\mu\rho} \\
[M^{\mu\nu}, x^\rho] &= i(x^\nu \eta^{\rho\mu} - x^\mu \eta^{\rho\nu}) + i\psi_\theta^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda}).
\end{aligned}
\tag{10}
$$

The function $\psi_\theta^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda})$ has the physical dimension of length and is antisymmetric in the first two indices. Inserting this ansatz into Condition 2

$$
\begin{aligned}
0 &= [[P^\mu, P^\nu], x^\lambda] + [[P^\nu, x^\lambda], P^\mu] + [[x^\lambda, P^\mu], P^\nu] \\
0 &= [[M^{\mu\nu}, P^\rho], x^\lambda] + [[P^\rho, x^\lambda], M^{\mu\nu}] + [[x^\lambda, M^{\mu\nu}], P^\rho] \\
0 &= [[M^{\mu\nu}, M^{\rho\sigma}], x^\lambda] + [[M^{\rho\sigma}, x^\lambda], M^{\mu\nu}] + [[x^\lambda, M^{\mu\nu}], M^{\rho\sigma}]
\end{aligned}
$$

and replacing the commutators $[P^\mu, P^\nu], [M^{\mu\nu}, P^\rho]$ and $[M^{\mu\nu}, M^{\rho\sigma}]$ by their right hand sides, we obtain

$$
\begin{aligned}
0 =\ & [\psi_\theta^{\mu\nu\lambda}, P^\rho] - [\chi_\theta^{\mu\nu\rho}, x^\lambda] \\
0 =\ & i[M^{\rho\sigma}, \psi_\theta^{\mu\nu\lambda}] - i[M^{\mu\nu}, \psi_\theta^{\rho\sigma\lambda}] + i[\phi_\theta^{\mu\nu\rho\sigma}, x^\lambda] \\
& -\eta^{\mu\rho}\psi_\theta^{\nu\sigma\lambda} + \eta^{\nu\rho}\psi_\theta^{\mu\sigma\lambda} - \eta^{\nu\sigma}\psi_\theta^{\mu\rho\lambda} + \eta^{\mu\sigma}\psi_\theta^{\nu\rho\lambda} \\
& -\eta^{\sigma\lambda}\psi_\theta^{\mu\nu\rho} + \eta^{\rho\lambda}\psi_\theta^{\mu\nu\sigma} - \eta^{\mu\lambda}\psi_\theta^{\rho\sigma\nu} + \eta^{\nu\lambda}\psi_\theta^{\rho\sigma\mu}.
\end{aligned}
\tag{11}
$$

8

Turning finally to Condition 3

$$
\begin{aligned}
0 &= \left[\left[P^\lambda, x^\mu\right], x^\nu\right] + \left[\left[x^\mu, x^\nu\right], P^\lambda\right] + \left[\left[x^\nu, P^\lambda\right], x^\mu\right] \\
0 &= \left[\left[M^{\rho\sigma}, x^\mu\right], x^\nu\right] + \left[\left[x^\mu, x^\nu\right], M^{\rho\sigma}\right] + \left[\left[x^\nu, M^{\rho\sigma}\right], x^\mu\right]
\end{aligned}
$$

and replacing again by the corresponding right hand sides we obtain the single equation

$$
0 = i\left[\psi_\theta^{\mu\nu\rho}, x^\sigma\right] - i\left[\psi_\theta^{\mu\nu\sigma}, x^\rho\right] - \eta^{\mu\rho}\theta^{\nu\sigma} + \eta^{\nu\rho}\theta^{\mu\sigma} + \eta^{\mu\sigma}\theta^{\nu\rho} - \eta^{\nu\sigma}\theta^{\mu\rho}. \tag{12}
$$

This final relation shows that the ansatz for $\psi^{\mu\nu\rho}$ can never be chosen to be zero or constant and such the coproduct of $M^{\mu\nu}$ is necessarily deformed. In the classical limit $\theta^{\mu\nu} \to 0$ we have $\phi_\theta^{\mu\nu\rho\sigma}(P^\gamma, M^{\kappa\lambda}) \to 0$, $\psi_\theta^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda}) \to 0$ and $\chi_\theta^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda}) \to 0$.

Now we have obtained all conditions for $U_\theta^\lambda(\mathfrak{p})$ as a representation on $\mathfrak{X}_\theta$. In the next subsection we find solutions $U_\theta^\lambda(\mathfrak{p})$ by making an appropriate ansatz for the deformed Lorentz generator $M^{\mu\nu}$.

2.2   THE COMPUTATION OF EXPLICIT SOLUTIONS $U_\theta^{(\lambda_1,\lambda_2)}(\mathfrak{p})$

In general the functions $\phi_\theta^{\mu\nu\rho\sigma}(P^\gamma, M^{\kappa\lambda})$, $\psi_\theta^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda})$ and $\chi_\theta^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda})$ can be considered as power series in $\theta^{\mu\nu}$, given by

$$
\begin{aligned}
\phi_\theta^{\mu\nu\rho\sigma}(P^\gamma, M^{\kappa\lambda}) &= \sum_{k=1}^\infty \phi_{\theta,k}^{\mu\nu\rho\sigma}(P^\gamma, M^{\kappa\lambda}) \\
\psi_\theta^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda}) &= \sum_{k=1}^\infty \psi_{\theta,k}^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda}) \\
\chi_\theta^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda}) &= \sum_{k=1}^\infty \chi_{\theta,k}^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda}).
\end{aligned}
$$

The index of summation k denotes the power of $\theta^{\mu\nu}$ in $\phi_{\theta,k}^{\mu\nu\rho\sigma}(P^\gamma, M^{\kappa\lambda})$, $\psi_{\theta,k}^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda})$ and $\chi_{\theta,k}^{\mu\nu\rho}(P^\gamma, M^{\kappa\lambda})$ respectively. We merely want to consider the most simple solutions to the set of equations (4), (11) and (12) and thus we restrict ourselves to the case that is linear in $\theta^{\mu\nu}$. If we account for the physical unities, we find that

$$
\begin{aligned}
\phi_\theta(P^\gamma, M^{\mu\nu}) = \phi_\theta(P^\gamma) &\sim \theta PP \\
\psi_\theta(P^\gamma, M^{\mu\nu}) = \psi_\theta(P^\gamma) &\sim \theta P \\
\chi_\theta(P^\gamma, M^{\mu\nu}) = \chi_\theta(P^\gamma) &\sim \theta PPP.
\end{aligned}
$$

Inserting this ansatz into the three conditions from the previous section generates the set of solutions in first order in θ. An alternative method that gives the same resuts is assuming the deformed Lorentz generator $M^{\mu\nu}$ to be of the following general form

$$
M^{\mu\nu} = x^\mu P^\nu - x^\nu P^\mu + \Lambda^{\mu\nu}. \tag{13}
$$

The function $\Lambda^{\mu\nu}$ has physical dimension 1 and is antisymmetric in its indices. It turns out in the next steps that any choice of $\Lambda^{\mu\nu}$ with these properties generates a valid solution of

$U_\theta^\lambda(\mathfrak{p})$. We now express the functions $\psi_\theta^{\mu\nu\rho}(P^\gamma)$, $\phi_\theta^{\mu\nu\rho\sigma}(P^\gamma)$ and $\chi_\theta^{\mu\nu\rho}(P^\gamma)$ in terms of $\Lambda^{\mu\nu}$

$$
\begin{aligned}
\chi_\theta^{\mu\nu\rho} &= -i[\Lambda^{\mu\nu}, P^\rho] \\
\psi_\theta^{\mu\nu\rho} &= \theta^{\mu\rho}P^\nu - \theta^{\nu\rho}P^\mu - i\left[\Lambda^{\mu\nu}, x^\rho\right] \\
\phi_\theta^{\mu\nu\rho\sigma} &= -\eta^{\mu\sigma}\Lambda^{\rho\nu} - \eta^{\nu\sigma}\Lambda^{\mu\rho} + \eta^{\mu\rho}\Lambda^{\sigma\nu} + \eta^{\nu\rho}\Lambda^{\mu\sigma} \\
&\quad +\theta^{\mu\rho}P^\nu P^\sigma - \theta^{\nu\rho}P^\mu P^\sigma - \theta^{\mu\sigma}P^\nu P^\rho + \theta^{\nu\sigma}P^\mu P^\rho \\
&\quad -i\left[\Lambda^{\mu\nu}, x^\rho\right]P^\sigma + i\left[\Lambda^{\mu\nu}, x^\sigma\right]P^\rho - i\left[x^\mu, \Lambda^{\rho\sigma}\right]P^\nu + i\left[x^\nu, \Lambda^{\rho\sigma}\right]P^\mu \\
&\quad +ix^\mu[\Lambda^{\rho\sigma}, P^\nu] - ix^\nu[\Lambda^{\rho\sigma}, P^\mu] + ix^\sigma[\Lambda^{\mu\nu}, P^\rho] - ix^\rho[\Lambda^{\mu\nu}, P^\sigma] \\
&\quad -i\left[\Lambda^{\mu\nu}, \Lambda^{\rho\sigma}\right].
\end{aligned} \tag{14}
$$

Inserting these expressions into the conditions (4), (11) and (12) results in

$$
\begin{aligned}
0 &= \left[[\Lambda^{\mu\nu}, \Lambda^{\rho\sigma}], P^\lambda\right] + \left[[\Lambda^{\rho\sigma}, P^\lambda], \Lambda^{\mu\nu}\right] + \left[[P^\lambda, \Lambda^{\mu\nu}], \Lambda^{\rho\sigma}\right] \\
0 &= \left[[\Lambda^{\mu\nu}, \Lambda^{\rho\sigma}], \Lambda^{\kappa\lambda}\right] + \left[[\Lambda^{\rho\sigma}, \Lambda^{\kappa\lambda}], \Lambda^{\mu\nu}\right] + \left[[\Lambda^{\kappa\lambda}, \Lambda^{\mu\nu}], \Lambda^{\rho\sigma}\right]
\end{aligned}
$$

$$
\begin{aligned}
0 &= [[\Lambda^{\mu\nu}, x^\sigma], P^\rho] \\
&= [[\Lambda^{\mu\nu}, x^\sigma], P^\rho] + [[x^\sigma, P^\rho], \Lambda^{\mu\nu}] + [[P^\rho, \Lambda^{\mu\nu}], x^\sigma] \\
0 &= \left[[\Lambda^{\mu\nu}, \Lambda^{\rho\sigma}], x^\lambda\right] + \left[[\Lambda^{\rho\sigma}, x^\lambda], \Lambda^{\mu\nu}\right] + \left[[x^\lambda, \Lambda^{\mu\nu}], \Lambda^{\rho\sigma}\right]
\end{aligned}
$$

$$
\begin{aligned}
0 &= [[\Lambda^{\mu\nu}, x^\rho], x^\sigma] + [[x^\sigma, \Lambda^{\mu\nu}], x^\rho] \\
&= [[\Lambda^{\mu\nu}, x^\rho], x^\sigma] + [[x^\sigma, \Lambda^{\mu\nu}], x^\rho] + [[x^\rho, x^\sigma], \Lambda^{\mu\nu}].
\end{aligned} \tag{15}
$$

Due to its physical dimension, the most simple structure of $\Lambda^{\mu\nu}$ is of the form

$$\Lambda^{\mu\nu} \sim \theta PP.$$

Obviously any choice of $\Lambda^{\mu\nu}$ of this kind leads to a solution for $U_\theta^\lambda(\mathfrak{p})$. Omitting a possible constant, we choose $\Lambda^{\mu\nu}$ to be

$$\Lambda^{\mu\nu} := \lambda_1 P_\alpha\left(\theta^{\mu\alpha}P^\nu - \theta^{\nu\alpha}P^\mu\right) + \lambda_2\eta_{\alpha\beta}P^\alpha P^\beta\theta^{\mu\nu}, \tag{16}$$

with real parameters λ_1, λ_2. Computing now the functions $\psi^{\mu\nu\rho}(P^\gamma)$, $\phi^{\mu\nu\rho\sigma}(P^\gamma)$ and $\chi^{\mu\nu\rho\sigma}(P^\gamma)$ with this expression, we obtain

$$
\begin{aligned}
\chi_\theta^{\mu\nu\rho} &= 0 \\
\psi_\theta^{\mu\nu\rho} &= (1-\lambda_1)\left(\theta^{\mu\rho}P^\nu - \theta^{\nu\rho}P^\mu\right) + \lambda_1 P_\alpha\left(\eta^{\mu\rho}\theta^{\nu\alpha} - \eta^{\nu\rho}\theta^{\mu\alpha}\right) - 2\lambda_2\theta^{\mu\nu}P^\rho \\
\phi_\theta^{\mu\nu\rho\sigma} &= (1-2\lambda_1)\left(\theta^{\mu\rho}P^\nu P^\sigma - \theta^{\nu\rho}P^\mu P^\sigma - \theta^{\mu\sigma}P^\nu P^\rho + \theta^{\nu\sigma}P^\mu P^\rho\right) \\
&\quad -\lambda_2\left(\theta^{\mu\rho}\eta^{\nu\sigma} - \theta^{\nu\rho}\eta^{\mu\sigma} - \theta^{\mu\sigma}\eta^{\nu\rho} + \theta^{\nu\sigma}\eta^{\mu\rho}\right)\eta_{\alpha\beta}P^\alpha P^\beta
\end{aligned} \tag{17}
$$

and by this we have finally found all solutions $U_\theta^{(\lambda_1, \lambda_2)}(\mathfrak{p})$. The solution $U_\theta^{(\frac{1}{2}, 0)}(\mathfrak{p})$ gives the classical relations for the Lorentz algebra but with deformed coproduct - as we shall see in the next section. This result was also obtained by alternative considerations [4], [21].

2.3   THE HOPF ALGEBRA STRUCTURE OF $U_\theta^{(\lambda_1, \lambda_2)}(\mathfrak{p})$

In this final part we discuss the Hopf algebra structure of $U_\theta^{(\lambda_1, \lambda_2)}(\mathfrak{p})$. Since the subalgebra of momentum operators P^μ is undeformed

$$\Delta(P^\mu) = P^\mu \otimes \mathbf{1} + \mathbf{1} \otimes P^\mu, \ \epsilon(P^\mu) = 0, \ S(P^\mu) = -P^\mu,$$

we merely focus on the corresponding properties for the deformed Lorentz generators $M^{\mu\nu}$. The necessary computations to prove the Hopf algebra axioms for $M^{\mu\nu}$ are straight forward, such that we limit ourselves to present the results and step through the necessary points without going into computational details. To ensure that counit and coproduct are algebra homomorphisms, they have to map the unit operator $\mathbf{1} \in U_\theta^{(\lambda_1,\lambda_2)}(\mathfrak{p})$ according to

$$
\begin{aligned}
\epsilon(\mathbf{1}) &= 1 \\
\Delta(\mathbf{1}) &= \mathbf{1} \otimes \mathbf{1}.
\end{aligned}
$$

From relation (9) we read of the coproduct of $M^{\mu\nu}$ to be

$$
\begin{aligned}
\Delta(M^{\mu\nu}) &= M^{\mu\nu} \otimes \mathbf{1} + \mathbf{1} \otimes M^{\mu\nu} - (1-\lambda_1)P_\alpha \otimes (\theta^{\mu\alpha}P^\nu - \theta^{\nu\alpha}P^\mu) \\
&\quad + \lambda_1\left(\theta^{\mu\alpha}P^\nu - \theta^{\nu\alpha}P^\mu\right) \otimes P_\alpha + 2\lambda_2\theta^{\mu\nu}\eta^{\alpha\beta}P_\alpha \otimes P_\beta.
\end{aligned} \tag{18}
$$

Choosing the counit of $M^{\mu\nu}$ by

$$
\epsilon(M^{\mu\nu}) = 0,
$$

it is easy to see that $U_\theta^{(\lambda_1,\lambda_2)}(\mathfrak{p})$ satisfies the axioms of a bialgebra by proving the coalgebra axioms presented in the first subsection, such as counit and coassociativity

$$
\begin{aligned}
(\epsilon \otimes \mathrm{id}) \circ \Delta(M^{\mu\nu}) = \quad \mathrm{id} \quad &= (\mathrm{id} \otimes \epsilon) \circ \Delta(M^{\mu\nu}) \\
(\Delta \otimes \mathrm{id}) \circ \Delta(M^{\mu\nu}) &= (\mathrm{id} \otimes \Delta) \circ \Delta(M^{\mu\nu}).
\end{aligned}
$$

To finally make $U_\theta^{(\lambda_1,\lambda_2)}(\mathfrak{p})$ a Hopf algebra, we need an antipode S for $M^{\mu\nu}$. We find that

$$
\mathrm{S}(M^{\mu\nu}) = -M^{\mu\nu} - (1-2\lambda_1)(\theta^{\mu\alpha}P^\nu - \theta^{\nu\alpha}P^\mu)P_\alpha + 2\lambda_2\theta^{\mu\nu}\eta_{\alpha\beta}P^\alpha P^\beta \tag{19}
$$

satisfies the axiom for the antipode

$$
m \circ (\mathrm{id} \otimes \mathrm{S}) \circ \Delta(M^{\mu\nu}) = \epsilon(M^{\mu\nu})\mathbf{1} = m \circ (\mathrm{S} \otimes \mathrm{id}) \circ \Delta(M^{\mu\nu}),
$$

where m represents the multiplication within $U_\theta^{(\lambda_1,\lambda_2)}(\mathfrak{p})$. We remark that the double application of the antipode map S on $M^{\mu\nu}$ is the identity operator

$$
\mathrm{S}^2 = \mathrm{id}.
$$

The coproduct $\Delta(M^{\mu\nu})$ and the antipode $S(M^{\mu\nu})$ for $\theta^{\mu\nu} \to 0$ converge to the undeformed case

$$
\Delta(m^{\mu\nu}) = m^{\mu\nu} \otimes 1 + 1 \otimes m^{\mu\nu}, \; S(m^{\mu\nu}) = -m^{\mu\nu},
$$

with $m^{\mu\nu} \in U(\mathfrak{p})$. Finally we have to ensure that coproduct and counit are algebra homomorphisms. Since the counit is trivial, the task reduces itself to satisfy the relations

$$
\begin{aligned}
[\Delta(M^{\mu\nu}), \Delta(P^\rho)] &= i\eta^{\mu\rho}\Delta(P^\mu) - i\eta^{\nu\rho}\Delta(P^\nu) \\
[\Delta(M^{\mu\nu}), \Delta(M^{\rho\sigma})] &= i\eta^{\mu\rho}\Delta(M^{\nu\sigma}) - i\eta^{\nu\rho}\Delta(M^{\mu\sigma}) + i\eta^{\nu\sigma}\Delta(M^{\mu\rho}) - i\eta^{\mu\sigma}\Delta(M^{\nu\rho}) \\
&\quad + i\Delta(\phi^{\mu\nu\rho\sigma})
\end{aligned}
$$

An easy computation shows that this is the case for all solutions $U_\theta^{(\lambda_1,\lambda_2)}(\mathfrak{p})$ that we have presented here.

3   CASIMIR OPERATORS AND SPACE INVARIANTS

In order to study field theoretic properties of the presented deformations $U^{(\lambda_1,\lambda_2)}(\mathfrak{p})$ we consider its central elements and spacetime invariants in this final section.

Since the algebra of momenta P^μ is undeformed, the d'Alembert operator $\Box = \partial_\mu \partial^\mu = -P_\nu P^\nu$ and thus the Klein-Gordon operator are those of the classical case.

Concerning the Pauli-Lubanski vector and the spacetime invariant the situation is changed. We present a deformed Pauli-Lubanski vector W^λ that transforms as a classical vector under the action of the Lorentz operators $M^{\mu\nu}$, such that its square is invariant under these operations. In order to obtain a spacetime invariant we find that the parameters λ_1 and λ_2 become dependent.

3.1   PAULI-LUBANSKI VECTOR

Since the commutation relations of the Lorentz generators $[M^{\mu\nu}, M^{\alpha\beta}]$ are deformed in general, the classical Pauli-Lubanski vector $\epsilon^{\lambda\kappa\rho\sigma} P_\kappa M_{\rho\sigma}$ does not transform as a classical vector anymore

$$
\begin{aligned}
[M^{\mu\nu}, \epsilon^{\lambda\kappa\rho\sigma} P_\kappa M_{\rho\sigma}] &= i\eta^{\mu\lambda} \epsilon^{\nu\kappa\rho\sigma} P_\kappa M_{\rho\sigma} - i\eta^{\nu\lambda} \epsilon^{\mu\kappa\rho\sigma} P_\kappa M_{\rho\sigma} \\
&\quad + 2i\lambda_2 (\theta^\mu_\alpha \epsilon^{\nu\alpha\lambda\rho} - \theta^\nu_\alpha \epsilon^{\mu\alpha\lambda\rho}) P_\rho P_\beta P^\beta.
\end{aligned}
$$

Moreover its square $W^\lambda W_\lambda$ turns out not to be invariant as well. We define the deformed Pauli-Lubanski vector by the following properties

$$
\begin{aligned}
\left[M^{\mu\nu}, W^\lambda \right] &= i\eta^{\mu\lambda} W^\nu - i\eta^{\nu\lambda} W^\mu \\
\left[M^{\mu\nu}, W_\lambda W^\lambda \right] &= 0 \\
\left[P^\mu, W_\lambda W^\lambda \right] &= 0,
\end{aligned}
\tag{20}
$$

and make an ansatz of the form $\epsilon^{\lambda\kappa\rho\sigma} P_\kappa M_{\rho\sigma} + (\epsilon\theta PPP)^\lambda$. We thus obtain the deformed Pauli-Lubanski vector to be

$$
W^\lambda = \epsilon^{\lambda\kappa\rho\sigma} P_\kappa M_{\rho\sigma} + \lambda_2 \epsilon^{\lambda\kappa\rho\sigma} \theta_{\kappa\rho} P_\sigma P_\alpha P^\alpha.
\tag{21}
$$

It is remakable that W^λ is independent of λ_1.

3.2   SPACE INVARIANTS

Concerning the spacetime invariant I we demand that it is merely an element of $\mathfrak{X}_\theta$. This is a strong requirement, since it becomes impossible to deform I in any way. On the other hand the coproduct of $M^{\mu\nu}$ and thus its action on $I = x^\rho x_\rho$ is necessarily deformed, as we stated in reference to relation (12). We obtain for the action of $M^{\mu\nu}$ on $I = x^\rho x_\rho$

$$
\begin{aligned}
M^{\mu\nu} \triangleright (x^\rho x_\rho) &= ([M^{\mu\nu}, x^\rho x_\rho]) \triangleright \mathbf{1} \\
&= (-\theta^{\mu\nu}(2\lambda_2 n + 4\lambda_1 - 2) - 4i\lambda_2 \theta^{\mu\nu} x^\rho P_\rho - 2i\lambda_1 (\theta^{\mu\rho} x^\nu P_\rho - \theta^{\nu\rho} x^\mu P_\rho) \\
&\quad - 2i(\lambda_1 - 1)(\theta^{\mu\rho} x_\rho P_\nu - \theta^{\nu\rho} x_\rho P_\mu)) \triangleright \mathbf{1} \\
&= -\theta^{\mu\nu}(2\lambda_2 n + 4\lambda_1 - 2),
\end{aligned}
\tag{22}
$$

where n denotes the dimension of spacetime. To ensure that

$$M^{\mu\nu} \rhd I = 0$$

we thus have to require that

$$\lambda_1 = \frac{1}{2}(1 - n\lambda_2), \tag{23}$$

and such the parameters λ_1 and λ_2 become dependent.

4 Conclusion

In this paper we have derived a set of deformations $U_\theta^{(\lambda_1,\lambda_2)}(\mathfrak{p})$ as representations on non-commutative spacetime algebras $\mathfrak{X}_\theta$. We have furthermore indicated how such deformations could possibly be constructed for arbitrarily given spacetime algebras.

At the moment it is still unclear whether the obtained solutions $U_\theta^{(\lambda_1,\lambda_2)}(\mathfrak{p})$ are related to each other. For physical reasons for example all solutions with $\lambda_2 = 0$ might turn out to be equivalent. Indeed, there are two hints in the present work that support this assumption. Firstly it is remarkable that the deformed Pauli-Lubanski vector is independent of λ_1, provided that $\lambda_2 = 0$. With respect to particle states this means that the notion of spin does not depend on deformations of this kind. Secondly, when the algebras $U_\theta^{(\lambda_1,0)}(\mathfrak{p})$ are represented on particle states the function $\phi^{\mu\nu\rho\sigma}$ can be treated as a constant. So the contribution of this part of the deformation is likely to be only a phase factor similar to that generated by Moyal-Weyl star products, and one may speculate whether λ_1 could be treated as a $U(1)$ gauge parameter.

We would like to develop the ideas presented in this paper into a method that yields deformations for any given noncommutative space. To this end it seems important to find a generating function for the general case, such as $\Lambda^{\mu\nu}$ for the special case of $U_\theta^{(\lambda_1,\lambda_2)}(\mathfrak{p})$.

Finally, since the functions $\phi^{\mu\nu\rho\sigma}$ and $\psi^{\mu\nu\rho}$ depend on each other it should be possible to derive the Hopf structure directly within the deformation procedure.

5 Acknowledgment

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